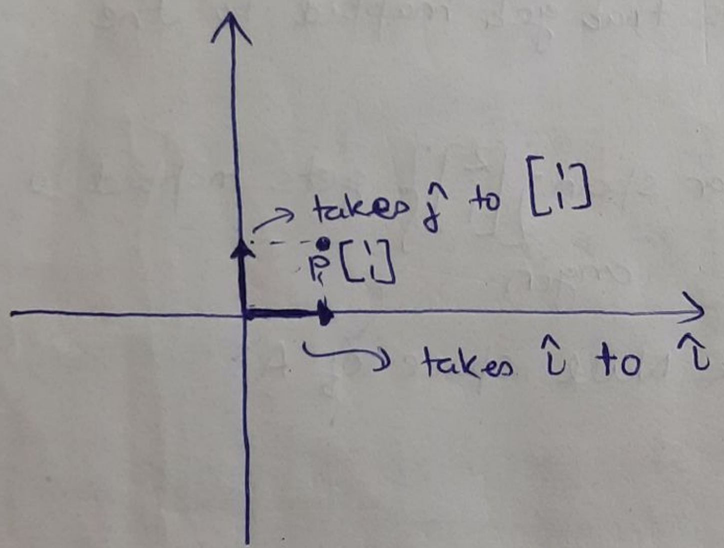
Lecture #2

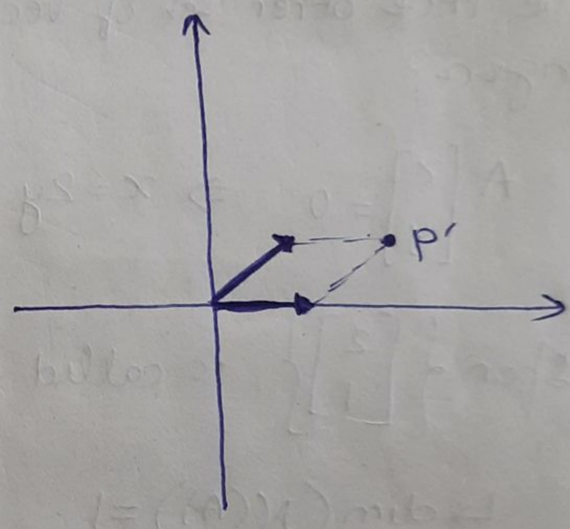
→ In the last class, we looked at linear transformations of vectors in a vector space.

→ In the process, we started viewing matrices as linear transformations, where columns of a matrix represent transformation of standard basis vectors.

Consider the transformation

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$





- What about the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?
  - ↳ Linear transformations warp the space such that parallel lines remain parallel, origin maps to origin and grid lines are equally spaced.
  - ↳ Of course, a transformation can invert the orientation.

- Example: Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \rightarrow \text{rank}(A) = 1$$

↓ # of linearly independent columns.

Each vector in the input space gets mapped to a straight line.

- Span of columns of a matrix is defined as the Column space or Range space of a Matrix. If all columns are linearly independent, then the matrix is full-rank.

- Looking back at the previous case:

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \rightarrow \text{A maps all input vectors to a straight line.}$$

↳ By definition A also maps origin to origin

- Are there other set of vectors that get mapped to the origin?

$$A\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow x = 2y \text{ or span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \text{ gets mapped to origin.}$$

$\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ is called the Null space of A.

↳ $\dim(\mathcal{N}(A)) = 1$

- $\text{Rank}(A) = \dim(\text{col space}(A)) = \dim(\text{row space}(A)) = 1$
- $\dim(\mathcal{N}(A)) = 1$

$$\boxed{\text{Rank}(A) + \dim(\mathcal{N}(A)) = 2 \text{ (size of matrix)}}$$

↳ Rank-Nullity theorem

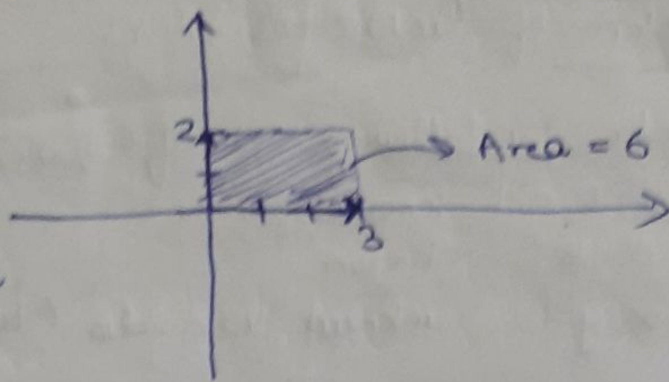
- Also note that $\mathcal{R}(A^T)$ is orthogonal complement of $\mathcal{N}(A)$.

- Determinant of a matrix:

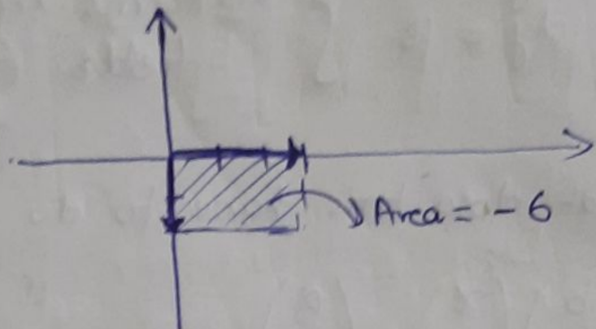
$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

"Area" of unit grid in transformed space

or volume of parallelepiped



$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$



$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

↳ signed area

In higher dimensions, it ~~do~~ represents the volume.

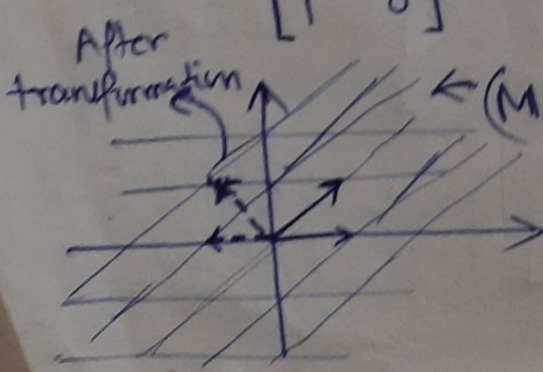
* Composition of Matrices:

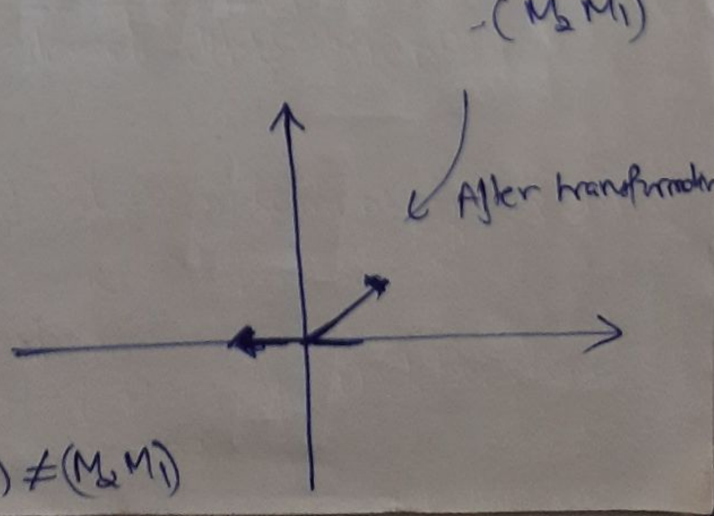
Consider two matrices M_1 and M_2

"Rotation" "Shear"

and look at their composite effect

$$M_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$





In general

$$(M_1 M_2) \neq (M_2 M_1)$$

* $\det(AB) = \det(A) * \det(B)$ why?

* Solving system of linear Equations.

Consider the following set of equations.

a, b, c, d known

α, β known

x, y unknown variables (to be determined)

$$\left.\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned}\right\} \Rightarrow$$ Forms the basis of various fields (One of the most fundamental problems)

↳ what does this have to do with matrices?

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{\vec{b}}$$

$A\vec{x} = \vec{b}$ (Find $\vec{x}$)

Q: Does such an $\vec{x}$ always exist?

Is $\vec{x}$ unique?

Properties of A and $\vec{b}$ that result in existence and uniqueness.

rank(A) may have something to do with it.

A: Recall $A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$

If $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ are linearly dependent, i.e. rank $(A) < 2$,

then the output $x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$ will be in span of $\begin{pmatrix} a \\ c \end{pmatrix}$.

eg: Let $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

Then $x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -2 \\ -4 \end{bmatrix} = (x-2y)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So, the vector $A\vec{x}$ is again in the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

So if one asks for a solution of the form

$$A\vec{x} = \vec{b} \quad \text{where } A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

then, there does not exist any solution $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $A\vec{x} = \vec{b}$

since $\vec{b}$ is not in the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

On the other hand, if $\vec{b}$ is in the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (for eg. $\vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$), then there are infinitely many combinations of x and y such that $A\vec{x} = \vec{b}$, since the only constraint is $x - 2y = 3$.

Hence, if a matrix loses rank, depending on $\vec{b}$, we may have
1. No solution
2. Infinitely many solutions.

* Eigenvectors and Eigenvalues

Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$.

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

i.e., vector $\hat{i}$ gets transformed to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

vector $\hat{j}$ gets transformed to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus, input vectors get transformed to a vector in different direction. However, there are certain vectors whose direction remains invariant under linear transformation.

Such vectors are called eigenvectors. If $\vec{v}$ is an eigenvector, then

$$A\vec{v} = \lambda\vec{v}$$

$\hookrightarrow$ Eigenvalue

How to find Eigenvectors and Eigenvalues:

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A-\lambda I)\vec{v} = \vec{0}$$

$\hookrightarrow$ eigenvectors lie in the null space of $(A-\lambda I)$.

For finding eigenvalues, $\det(A-\lambda I) = 0$.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow A-\lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{bmatrix}$

$\Rightarrow \det(A-\lambda I) = 0 \Rightarrow \lambda(\lambda-1)-2 = 0 \Rightarrow \lambda = -1, 2$

When $\lambda = -1$:

$A-\lambda I = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

When $\lambda = 2$:

$A-\lambda I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$